ON INTEGRATION IN VECTOR SPACES*

BY

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Introduction. Several authors ([3]–[10], inclusive; [15])† have already given generalized Lebesgue integrals for functions $x(s)$ whose values lie in a Banach space (B-space) $\mathfrak{X}$.‡ In the following pages another definition,§ based on the linear functionals over $\mathfrak{X}$ and on the ordinary Lebesgue integral,|| will be given and its properties and relationships to other integrals discussed.

We consider functions $x(s)$ defined to a B-space $\mathfrak{X}=[x]$ from an abstract space $S=[s]$ possessing both a σ-field Σ of "measurable" sets having S as an element and a non-negative bounded c.a. (completely additive) "measure" function $\alpha(E)$ defined over Σ. Notational conventions are as follows: $\overline{\mathfrak{X}}$ denotes the B-space conjugate to $\mathfrak{X}$ ([1], p. 188), $f=f(x)$, $g=g(x)$, $\cdots$ denote elements of $\overline{\mathfrak{X}}$, and $F=F(f)$, $G=G(f)$, $\cdots$ elements of $\overline{\overline{\mathfrak{X}}}$, the conjugate space of $\overline{\mathfrak{X}}$; and for real-valued functions Greek letters will be used. When the abstract functions $x(s)$, $y(s)$, $\cdots$, or the real functions $\phi(s)$, $\psi(s)$, $\cdots$ are considered as elements of a functional space, they will sometimes be written $x(:)$, $y(:)$ or $\phi(:)$, $\psi(:)$. The "zero" element of $\mathfrak{X}$ will be denoted by θ.

The contents of the paper may be outlined as follows. The first section compares measurability of functions under the strong and weak topologies of $\mathfrak{X}$. The second defines the ($\mathfrak{X}$) integral and utilizes an essential lemma, first stated by Orlicz, to prove the complete additivity and absolute continuity of the integral. The linear operations from L^p to $\mathfrak{X}$ and from $\overline{\mathfrak{X}}$ to $\overline{L^p}$ defined by integrable functions are investigated in §3, and these results are expressed in terms of real-valued kernels in §7. Approximation and convergence theorems occupy §4 and lead to §5 and the relationships between the ($\mathfrak{X}$) integral and the integrals given by other definitions. A few remarks on completely continuous transformations and on differentiation account for §§6 and 8, respectively, and four examples form §9. In conclusion a few open questions are cited.

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† Numerals in brackets refer to the list of references at the end of the paper.

‡ See [1]. The terminology used will be that of Banach's book.

§ Suggested to the author by E. J. McShane. The definition is also due to Dunford, *Uniformity in linear spaces*, these Transactions, vol. 44 (1938), pp. 305–356. Also see [10], p. 50, footnote.

The author wishes to take this opportunity to express his gratitude to Professor McShane for for his most kind and helpful criticism during the writing of this paper.

|| From the definition it will be evident that any integral for real-valued functions can be generalized in the same fashion.

For reference purposes the following well known theorems concerning B-spaces will be listed:*

A. *Given a linear functional (an additive, continuous, real-valued function) $g(y)$ defined in a linear subset $\mathfrak{Y}$ of $\mathfrak{X}$, there exists a linear functional $f(x)$ defined in $\mathfrak{X}$ and such that $f(y)=g(y)$, for y in $\mathfrak{Y}$, and $\|f\|=\|g\|$ where the norm of f is taken over $\mathfrak{X}$ and that of g over $\mathfrak{Y}$.*

B. *For every x_0 in $\mathfrak{X}$ there is a linear functional $f_0(x)$ defined over $\mathfrak{X}$ and such that $|f_0(x_0)|=\|x_0\|$, and $\|f_0\|=1$.*

C. *Given a sequence $\{x_n\}$ of elements of $\mathfrak{X}$, the closed linear hull (the closure of the set of all finite linear combinations) of the x_n's is a separable sub-B-space of $\mathfrak{X}$.*

1. **Measurable and weakly measurable functions.** A function $x(s)$ from S to $\mathfrak{X}$ will be called a *step-function* if and only if $x(s)$ is constant on each of a finite number of disjoint measurable sets whose sum is S. As in Bochner [4], $x(s)$ is called *measurable* provided it is a.e. (almost everywhere) the strong limit of a sequence of step-functions; a measurable function taking only a countable number of distinct values will be said to be *countably-valued.* If there exists a separable sub-B-space $\mathfrak{Y}$ and a set S' such that $\alpha(S')=\alpha(S)$ and s in S' implies $x(s)$ in $\mathfrak{Y}$, then $x(s)$ is *separably-valued.*

It is evident that a measurable function $x(s)$ is also *weakly measurable*, that is, $f(x(s))$ is measurable ([2], p. 251) for every f in $\overline{\mathfrak{X}}$. The relationship between measurability and weak measurability is given in the following theorem:

THEOREM 1.1. *A necessary and sufficient condition that $x(s)$ be measurable is that it be weakly measurable and separably-valued.*

If $x(s)$ is the limit a.e. of step-functions $x_n(s)$, then almost all of its values lie in the separable closed linear hull (see C) of the denumerable set of values of the functions $x_n(s)$; thus $x(s)$ is separably-valued.

Now suppose $x(s)$ is weakly measurable and, with no loss of generality, that all the values of $x(s)$ lie in a separable subspace $\mathfrak{Y}$.

The first conclusion is that $\|x(s)\|$ is measurable. Let $\{g_i\}$ be weakly dense ([1], p. 123) in the surface $\mathfrak{S}$ of the unit sphere of $\overline{\mathfrak{Y}}$. If $A\equiv S[\|x(s)\|\leqq a]$, and $A_g\equiv S[|g(x(s))|\leqq a]$ for each g in $\mathfrak{S}$, then $A=\prod A_g$, where the product is taken over $g\in\mathfrak{S}$. For if $\|x(s)\|\leqq a$, then for any g in $\mathfrak{S}$ we have $\|g\|=1$ and hence $|g(x(s))|\leqq\|x(s)\|\leqq a$; and for the fixed element $x(s)$ of $\mathfrak{Y}$ there is, by B, a $\bar{g}$ in $\mathfrak{S}$ such that $\|x(s)\|=|\bar{g}(x(s))|$. Thus $A=\prod A_g\subset\prod_i A_{g_i}$, where the first

* The first two are Theorems 2 and 3, [1], p. 55.

product is taken over $g \epsilon \mathfrak{S}$. The sequence $\{g_i\}$ being weakly dense, for any g in $\mathfrak{S}$ there is a subsequence $\{g_i'\}$ such that $g_i'(x) \to g(x)$ for every x in $\mathfrak{X}$; thus if s is in $\prod A_{g_i}$, then $|g(x(s))| = \lim_i |g_i'(x(s))| \leqq a$ for each g in $\mathfrak{S}$, so that $A = \prod A_g = \prod_i A_{g_i}$, where the first product is taken over $g \epsilon \mathfrak{S}$. From A and the hypotheses on $x(s)$, each set A_{g_i} is measurable, which implies measurability for $\|x(s)\|$.

If each $x(s)$ is covered with an open $1/n$-sphere in separable $\mathfrak{Y}$, by Lindelöf's theorem a countable set $\mathfrak{Y}_{in}$, $(i=1, 2, \cdots, j, \cdots)$, of these spheres contains the set of functional values of $x(s)$. If x_{in} is the center of $\mathfrak{Y}_{in}$, then $x(s) - x_{in}$ is weakly measurable and has all its values in $\mathfrak{Y}$, so that $\|x(s) - x_{in}\|$ is measurable. Hence the set E_{in} composed of all s having $x(s)$ in $\mathfrak{Y}_{in}$ is a measurable set, and $\sum_i E_{in} = S$. If $x_n(s) = x_{in}$ for s in $E_{in}^* \equiv E_{in} - \sum_{j=1}^{i-1} E_{jn}$, then $\|x(s) - x_n(s)\| < 1/n$ for all s, since $\sum_i E_{in}^* = S$. Since the sets E_{jn} and E_{in}^* are measurable, $x_n(s)$ is constant on each of a countable number of disjoint measurable sets whose sum is S; $x_n(s)$ is therefore measurable, hence $x(s)$, the uniform limit of $\{x_n(s)\}$, is also measurable.

Corollary 1.11. *If $\mathfrak{X}$ is separable, measurability is equivalent to weak measurability.*

Corollary 1.12. *A function $x(s)$ is measurable if and only if it can be approximated uniformly, except possibly on a set of measure zero, by countably-valued functions.*

Corollary 1.13. *If $\{x(s)\}$ is a sequence of measurable functions converging weakly a.e. to $x(s)$, then $x(s)$ is measurable.*

For if $\mathfrak{Y}_n$ is a separable sub-B-space containing almost all the values of $x_n(s)$, let $\mathfrak{Y}$ be the separable closed linear hull of a denumerable set dense in $\sum_n \mathfrak{Y}_n$. If $x_n(s) \to x(s)$ weakly at the point s, then ([1], p. 134) some sequence of linear combinations of the elements $\{x_n(s)\}$ converges strongly to $x(s)$ and $x(s)$ is therefore in $\mathfrak{Y}$. Since this statement holds for almost all s, and since $x(s)$ is obviously weakly measurable, the above theorem gives the desired conclusion.

1.14. Suppose that S is euclidean, R_0 is a fixed elementary figure in S, and α is the Lebesgue measure function. If $X(R)$ is defined to $\mathfrak{X}$ from the elementary figures in R_0, then $X(R)$ is *weakly differentiable* at a point s in R_0 if there exists an element $x(s)$ in $\mathfrak{X}$ such that for every f in $\overline{\mathfrak{X}}$

$$f(x(s)) = \lim_{\alpha(I) \to 0} f\left(\frac{X(I)}{\alpha(I)}\right),$$

where I is an arbitrary cube containing s.

THEOREM 1.2. *If $X(R)$ is weakly differentiable a.e. in R_0 to $x(s)$, then $x(s)$ is measurable.*

Let I_0 be a cube containing R_0, and for each n let Π_n be a partition of I_0 into non-overlapping non-degenerate subcubes, with Π_{n+1} a repartition of Π_n and $\lim_{n\to\infty}$ (norm Π_n) $=0$. If s in R_0 is interior to a cube I_{jn} in Π_n, and if $R_0 \supset I_{jn}$, define

$$x_n(s) = \frac{X(I_{jn})}{\alpha(I_{jn})};$$

otherwise set $x_n(s)=\theta$. Each $x_n(s)$ is measurable, and by hypothesis $x_n(s)$ converges weakly to $x(s)$ for almost all s in R_0; from Corollary 1.13 this implies measurability for $x(s)$.

2. **Definition and properties of the integral.** We make the following definition:

DEFINITION 2.1. *A function $x(s)$ from S to $\mathfrak{X}$ is $(\mathfrak{X})$ integrable over measurable E if and only if there exists an x_E in $\mathfrak{X}$ such that*

$$f(x_E) = \int_E f(x(s))d\alpha$$

for every f in $\overline{\mathfrak{X}}$, wherein the integral on the right is the Lebesgue. By definition*

$$\int_E x(s)d\alpha = x_E.$$

A function $x(s)$ is integrable or $(\mathfrak{X})$ integrable if and only if it is $(\mathfrak{X})$ integrable over E for every measurable E.†

2.11. Immediate consequences of the definition are: (1) the integral is single-valued; (2) it is a linear function of the integrand; (3) if $\phi(s)$ is finite for every s and is Lebesgue integrable, and if $x(s)=x_0$, a constant, then $\phi(s)x(s)$ is defined and integrable and $\int_E\phi(s)x(s)d\alpha = x_0\cdot\int_E\phi(s)d\alpha$ (in particular, $\int_E x_0 d\alpha = x_0\cdot\alpha(E)$); (4) if $x(s)$ is integrable and $y(s)=x(s)$ a.e., then $y(s)$ is integrable and $\int_E y(s)d\alpha = \int_E x(s)d\alpha$ for every measurable E; (5) if $\mathfrak{X}$ is the space of reals, this definition coincides with the Lebesgue integral.

THEOREM 2.2. *If $x(s)$ from S to $\mathfrak{X}$ is integrable, and if $y=U(x)$ is a linear operation from $\mathfrak{X}$ to $\mathfrak{Y}$, then $y(s)\equiv U(x(s))$ is integrable and $U(\int_E x(s)d\alpha) = \int_E U(x(s))d\alpha$.*

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* [2], p. 247; [17].

† This definition is analogous to that of I. Gelfand [15]. Gelfand, however, considers functions defined from a linear interval to $\overline{\mathfrak{X}}$.

If any g in $\bar{\mathfrak{Y}}$ is given, the identity $g(y)=g(U(x))=f(x)$ defines an element f of $\bar{\mathfrak{X}}$. By hypothesis the function $f(x(s))=g(y(s))$ is Lebesgue integrable over any measurable E, and

$$f\left(\int_E x(s)d\alpha\right) = \int_E f(x(s))d\alpha = \int_E g(y(s))d\alpha$$

or

$$g\left(U\left(\int_E x(s)d\alpha\right)\right) = \int_E g(y(s))d\alpha.$$

This implies the conclusions of the theorem.

2.21. If (J) is an integral definition for functions $x(s)$ from S to $\mathfrak{X}$ that reduces to the Lebesgue when $\mathfrak{X}$ is the space of reals, and if the (J) integral has the above property of Theorem 2.2, it is clear, on letting $\mathfrak{Y}$ be the real numbers, that every $x(s)$ that is (J) integrable is also integrable and to the same value. Thus the ($\mathfrak{X}$) integral includes both that of Bochner ([9], p. 475) and that of G. Birkhoff ([6], p. 371). From this second inclusion it follows that an ($\mathfrak{X}$) integral is not always of bounded variation. However (Theorems 2.4 and 2.5), it is completely additive and, in the sense of Saks, absolutely continuous as a function of measurable sets.

2.3. If $\sum_n x_n$ is a series of elements of $\mathfrak{X}$, and if every subseries of $\sum_n x_n$ is convergent, then $\sum_n x_n$ is said to be *unconditionally convergent* ([12], p. 33); a function $X(E)$ from Σ to $\mathfrak{X}$ is *completely additive* ([6], p. 365) if and only if for every sequence $\{E_n\}$ of disjoint measurable sets $\sum_n X(E_n)$ is unconditionally convergent and $\sum_n X(E_n)=X(\sum_n E_n)$. A series $\sum_n x_n$ is said to have *property* (O) of Orlicz ([11], p. 244, Theorem 2) if and only if every subseries is weakly convergent to an element of $\mathfrak{X}$.

The next lemma and theorem were proved by Orlicz in [11] (Theorem 2) for the case $\mathfrak{X}$ weakly complete; the general theorem is credited by Banach ([1], p. 240) to the same author without proof or reference. Since we know no published proof to which to refer, and since the lemma is fundamental for our purposes, we include the following demonstration.*

LEMMA 2.31. *If $\sum_n x_n$ has property (O), then given $\epsilon>0$ there exists an N_ϵ such that f in $\bar{\mathfrak{X}}$ and $\|f\|=1$ implies $\sum_{n=N_\epsilon}^{\infty}|f(x_n)|<\epsilon$; hence $\|x_n\|\rightarrow 0$.*

Let $\mathfrak{Y}$ be the separable closed linear hull of the x_n's. To prove the first part it is sufficient to show that if $\{g_i\}$ is a weakly convergent sequence of functionals over $\mathfrak{Y}$ converging weakly to g_0, then in the space l^1 of absolutely convergent series the norms of the elements $\lambda_i=\{g_i(x_n)-g_0(x_n)\}$ converge to 0. For if there exist an $\epsilon_0>0$, a sequence $\{g_i\}$ in $\bar{\mathfrak{Y}}$, and integers $\{N_i\}$ such

* This lemma has been generalized by Dunford, *Uniformity in linear spaces*, loc. cit.

that $\|g_i\| \leqq 1$, and $N_i \to \infty$, and if

$$\sum_{n=N_i}^{\infty} | g_i(x_n) | \geqq \epsilon_0 > 0, \qquad \text{for all } i, \tag{1}$$

then since $\mathfrak{Y}$ is separable and $\|g_i\| \leqq 1$, we may suppose ([1], p. 123, Theorem 3) that $\{g_i\}$ is a weakly convergent sequence of functionals over $\mathfrak{Y}$, and hence that there is a g_0 such that $\{g_i\}$ converges weakly to g_0. But $\sum x_n$ has property (O), and for $i=0, 1, 2, \cdots$, the functional g_i is extensible from $\mathfrak{Y}$ to $\mathfrak{X}$; thus $\sum_{n=1}^{\infty} | g_i(x_n)| < \infty$, and if $\|\lambda_i\|_{l^1} \to 0$, the inequality (1) is contradicted and the first part of the lemma established.

To prove that $\|\lambda_i\| \to 0$ it is necessary and sufficient that to show ([1], pp. 138–139) $d(\lambda_i) \to 0$ for every linear functional $d(\lambda)$ defined over l^1 for which the associated bounded sequence $\{d_n\}$ is composed only of 0's, +1's, and −1's. If $d(\lambda)$ is such a functional, then by property (O)

$$\begin{aligned} \sum_n d_n f(x_n) &= \sum_{d_n=+1} f(x_n) - \sum_{d_n=-1} f(x_n) \\ &= f(x^+) - f(x^-) \\ &= f(x_0) \end{aligned} \tag{2}$$

for every f in $\overline{\mathfrak{X}}$. Then the sequence $\{\sum_{i=1}^{n} d_i x_i\}$ converges weakly to x_0, so that x_0 must be the strong limit of certain linear combinations of the elements $\{\sum_{i=1}^{n} d_i x_i\}$; hence x_0 is in $\mathfrak{Y}$.

From A and the fact that x_0 is in $\mathfrak{Y}$ it follows that (2) must hold for any g in $\overline{\mathfrak{Y}}$. This implies

$$\begin{aligned} d(\lambda_i) &\equiv \sum_{n=1}^{\infty} d_n \left[g_i(x_n) - g_0(x_n) \right] \\ &= g_i(x_0) - g_0(x_0) \end{aligned}$$

for each i. But $g_i(x) \to g_0(x)$ for every x in $\mathfrak{Y}$ including x_0; hence $d(\lambda_i) \to 0$.

The fact that $\|x_n\| \to 0$ follows immediately on applying B to the result just established.

THEOREM 2.32. *Unconditional convergence is equivalent to property* (O).

Unconditional convergence implies property (O), since convergence to an element implies weak convergence to that element.

If $\sum_n x_n$ is not unconditionally convergent, there is a non-convergent subseries. By inserting parentheses this subseries can be so "grouped" that the resulting series $\sum_m y_m$ has lim sup $\|y_m\| > 0$. If $\sum_n x_n$ has property (O), so does every subseries and every finite grouping of a subseries; then by the lemma $\|y_m\| \to 0$, and we have a contradiction.

THEOREM 2.4. *If $x(s)$ is integrable, then*

$$X(E) = \int_E x(s)d\alpha$$

is a completely additive function of measurable sets.

Let $\{E_n'\}$ be disjoint measurable sets. From the integrability of $x(s)$ and the complete additivity of the Lebesgue integral we have

$$f(X(E')) = \int_{E'} f(x(s))d\alpha = \sum_n \int_{E_n'} f(x(s))d\alpha$$
$$= \sum_n f\left(\int_{E_n'} x(s)d\alpha\right) = \sum_n f(X(E_n')),$$

where $E' = \sum_n E_n'$ and f in $\bar{\mathfrak{X}}$ is arbitrary. If any sequence $\{E_n\}$ of disjoint measurable sets is given, this must hold for every subsequence $\{E_n'\}$; hence $\sum_n X(E_n)$ has property (O) and is therefore unconditionally convergent by Theorem 2.32.

THEOREM 2.5.* *If $x(s)$ is integrable, then*

$$X(E) = \int_E x(s)d\alpha$$

is absolutely continuous.

If not there exist an $\epsilon > 0$ and sequences $\{E_i\}$, $\{f_i\}$ such that $\alpha(E_i) \to 0$, $\|f_i\| = 1$, and

$$\left|\int_{E_i} f_i(x(s))d\alpha\right| = \left|f_i\left(\int_{E_i} x(s)d\alpha\right)\right| = \left\|\int_{E_i} x(s)d\alpha\right\| \geqq \epsilon.$$

Let $\phi(s) = \text{l.u.b.}_i \, |f_i(x(s))| \leqq \|x(s)\|$, and let $M_n = S[n-1 \leqq \phi(s) < n]$ for each integer $n > 0$. The sets $\{M_n\}$ are disjoint and measurable, $\sum_n M_n = S$, and s in M_n implies $|f_i(x(s))| < n$ for all i.

Since $E_1 = \sum_n E_1 M_n$ and $X(E)$ is completely additive,

$$\epsilon \leqq \left\|\int_{E_1} x(s)d\alpha\right\| = \left\|\sum_{n=1}^{\infty} \int_{E_1 \cdot M_n} x(s)d\alpha\right\|;$$

there must then be an integer $n_1 > n_0 = 0$ such that

$$\left\|\sum_{n=n_0+1}^{n_1} \int_{E_1 \cdot M_n} x(s)d\alpha\right\| > \frac{\epsilon}{2} \cdot$$

* The method of proof here was suggested by E. J. McShane.

Suppose there exist sets $E_{i_1}, \cdots, E_{i_k}$ from $\{E_i\}$ and integers $n_0 < n_1 < \cdots < n_k$ such that

$$\left\| \sum_{n=n_{j-1}+1}^{n_j} \int_{E_{i_j}\cdot M_n} x(s)d\alpha \right\| > \frac{\epsilon}{2}$$

for $j=1, 2, \cdots, k$. Since $\alpha(E_i)\to 0$ there is an i_{k+1} such that $\alpha(E_{i_{k+1}}) < \epsilon/2n_k$. Then

$$\begin{aligned}
\epsilon &\leqq \left\| \int_{E_{i_{k+1}}} x(s)d\alpha \right\| = \left| \int_{E_{i_{k+1}}} f_{i_{k+1}}(x(s))d\alpha \right| \\
&\leqq \left| \sum_{n=1}^{n_k} \int_{E_{i_{k+1}}\cdot M_n} f_{i_{k+1}}(x(s))d\alpha \right| + \left| \sum_{n=n_k+1}^{\infty} \int_{E_{i_{k+1}}\cdot M_n} f_{i_{k+1}}(x(s))d\alpha \right| \\
&\leqq n_k\cdot\alpha(E_{i_{k+1}}) + \left| \sum_{n=n_k+1}^{\infty} \int_{E_{i_{k+1}}\cdot M_n} f_{i_{k+1}}(x(s))d\alpha \right| \\
&< \frac{\epsilon}{2} + \left| \sum_{n=n_k+1}^{\infty} \int_{E_{i_{k+1}}\cdot M_n} f_{i_{k+1}}(x(s))d\alpha \right|;
\end{aligned}$$

hence there clearly exists an $n_{k+1} > n_k$ such that

$$\left| \sum_{n=n_k+1}^{n_{k+1}} \int_{M_n\cdot E_{i_{k+1}}} f_{i_{k+1}}(x(s))d\alpha \right| > \frac{\epsilon}{2}.$$

Then, if one sets

$$G_{k+1} = \sum_{n=n_k+1}^{n_{k+1}} E_{i_{k+1}}\cdot M_n,$$

it follows that

$$\left\| \int_{G_{k+1}} x(s)d\alpha \right\| \geqq \left| \int_{G_{k+1}} f_{i_{k+1}}(x(s))d\alpha \right| > \frac{\epsilon}{2}\cdot$$

This completes the induction; thus there exist sequences $\{n_k\}$, $\{i_k\}$ of integers such that $n_k < n_{k+1}$, and if G_{k+1} is defined as above, for $k=0, 1, 2, \cdots$,

$$\left\| \int_{G_{k+1}} x(s)d\alpha \right\| > \frac{\epsilon}{2}\cdot$$

But the sets G_k are disjoint and measurable, so that $\sum_{k=1}^{\infty}\int_{G_k} x(s)d\alpha = \int_{\Sigma G_k} x(s)d\alpha$, whence $\left\|\int_{G_k} x(s)d\alpha\right\| \to 0$, a contradiction.

COROLLARY 2.51. *If $x(s)$ is integrable, then for any bounded set $\mathfrak{M}=[f]$ of elements of $\overline{\mathfrak{X}}$ the integrals $\int_E |f(x(s))|d\alpha$ are equi-absolutely continuous.*

If $\|f\| \leqq K$, we have $|\int_E f(x(s))d\alpha| = |f(\int_E x(s)d\alpha)| \leqq \|f\| \cdot \|\int_E x(s)d\alpha\| \leqq K \cdot \|\int_E x(s)d\alpha\| < \epsilon$ for $\alpha(E)$ less than a suitably chosen δ. Thus if $E_f^+ \equiv E[f(x(s)) \geqq 0]$ and $E_f^- = E - E_f^+$, we have

$$\int_E | f(x(s)) | \, d\alpha = \left| \int_{E_f^+} f(x(s))d\alpha \right| + \left| \int_{E_f^-} f(x(s))d\alpha \right| < 2\epsilon$$

for $\alpha(E) < \delta$, since $\alpha(E)$ is non-negative and $E_f^+ + E_f^- = E$.

3. **Operations defined by integrable functions.** In this section S will be the closed interval $[0, 1]$ and $\alpha(E)$ the Lebesgue measure function. According to custom L^p, $(1 \leqq p < \infty)$, will denote the class of real-valued measurable functions $\phi(s)$ defined over S and having $|\phi(s)|^p$ (L) integrable, and L^∞ the space of the real-valued essentially bounded and measurable functions defined over S. It is well known that L^p forms a B-space when $\|\phi\| = (\int_0^1 |\phi(s)|^p ds)^{1/p}$ for $1 \leqq p < \infty$ and $\|\phi\| = \text{ess. sup.}_S \, |\phi(s)|$ for $p = \infty$. There is no loss of generality in supposing that every element $\phi(:)$ of L^p has no infinite functional values.

Definition 3.1. *If $1 \leqq p' \leqq \infty$, then $\mathfrak{L}^{p'}(\mathfrak{X})$ is, for a given $\mathfrak{X}$, the class of all functions $x(s)$ from S to $\mathfrak{X}$ having $f(x(:))$ in $L^{p'}$ for every f in $\overline{\mathfrak{X}}$;* if $p' = \infty$, the added condition*

$$\underset{\|f\|=1}{\text{l.u.b.}} \, (\underset{S}{\text{ess.sup.}} \, | f(x(s)) |) < \infty$$

must be satisfied by each $x(:)$ in $\mathfrak{L}^\infty(\mathfrak{X})$; this l.u.b. *is denoted by $((x))_\infty$. The class $\mathfrak{L}_0^{p'}(\mathfrak{X})$ is defined to be the subclass of $\mathfrak{L}^{p'}(\mathfrak{X})$ composed of integrable functions.*

3.11. The class $\mathfrak{L}^\infty(\mathfrak{X})$ always includes the weakly measurable and essentially bounded functions, and $((x))_\infty \leqq \text{ess. sup.}_S \, \|x(s)\|$. A partial converse is that if the separably valued function $x(:)$ is in $\mathfrak{L}^\infty(\mathfrak{X})$, then $x(s)$ is not only measurable (by Theorem 1.1) but also essentially bounded. Suppose $x(s)$ is in $\mathfrak{Y}$, a separable subspace, for almost all s, and that $x(:)$ is in $\mathfrak{L}^\infty(\mathfrak{X})$. Let $\{g_i\}$ be weakly dense in the surface of the unit sphere of $\overline{\mathfrak{Y}}$, and let A be the set of values of s such that $\|x(s)\| > ((x))_\infty$ and A_i the set such that $|g_i(x(s))| > ((x))_\infty$. If $A' = S - A$, $A_i' = S - A_i$, then by the proof of Theorem 1.1 we have $A' = \prod_i A_i'$ and hence $A = \sum A_i$. Since $\alpha(A_i) = 0$, this implies that $x(s)$ is essentially bounded and that $\text{ess. sup.}_S \, \|x(s)\| = ((x))_\infty$.

Thus if $\mathfrak{X}$ is separable, $\mathfrak{L}^\infty(\mathfrak{X})$ is the class $S_\infty(\mathfrak{X})$ ([9], p. 474) composed of essentially bounded and measurable functions, that is, essentially bounded Bochner integrable functions. Since $\mathfrak{L}^\infty(\mathfrak{X}) \supset \mathfrak{L}_0^\infty(\mathfrak{X}) \supset S_\infty(\mathfrak{X})$, we have, for separable $\mathfrak{X}$,

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* This is due to Dunford (cf. [8]) and constitutes a direct generalization of the $(\mathfrak{X})$ integral.

$$S_\infty(\mathfrak{X}) = \mathfrak{L}^\infty(\mathfrak{X}) = \mathfrak{L}_0^\infty(\mathfrak{X}).$$

THEOREM 3.2. *If $x(:)$ is in $\mathfrak{L}^{p'}(\mathfrak{X})$, $(1 \leqq p' \leqq \infty)$, the operation*

$$U(f) = f(x(:))$$

from $\overline{\mathfrak{X}}$ to $L^{p'}$ is linear, and

$$\|U\| = \underset{\|f\|=1}{\text{l.u.b.}} \left(\int_0^1 | f(x(s)) |^{p'} ds \right)^{1/p'}, \qquad 1 \leqq p' < \infty,$$

$$\|U\| = ((x))_\infty, \qquad p' = \infty.$$

The additivity of U is trivial. In case $p' = \infty$

$$\|U(f)\| = \underset{S}{\text{ess. sup.}} \, | f(x(s)) | \leqq \|f\| \cdot ((x))_\infty,$$

and U is linear. If $p' < \infty$ and $f_n \to f_0$ in $\overline{\mathfrak{X}}$, then $f_n(x(s)) \to f_0(x(s))$ for every s, and by Fatou's lemma

$$\liminf_n \int_0^1 | f_n(x(s)) |^{p'} ds \geqq \int_0^1 | f_0(x(s)) |^{p'} ds$$

or $\liminf_n \|U(f_n)\| \geqq \|U(f_0)\|$, and U is again linear. The remainder of the theorem follows from the fact that $\|V\| = \text{l.u.b.}_{\|y\|=1} \|V(y)\|$ for an arbitrary linear operation $V(y)$.

DEFINITION 3.21. *In the linear space $\mathfrak{L}^{p'}(\mathfrak{X})$ the norm of $x(:)$ is $((x))_{p'} = \|U\|$.*

3.22. In the linear normed subspace $\mathfrak{L}_0^1(\mathfrak{X})$ a sequence $\{x_n\}$ is a Cauchy sequence if and only if $\text{l.u.b.}_E \|\int_E (x_m - x_n) ds\| \to 0$.* For

$$\underset{E}{\text{l.u.b.}} \left\| \int_E (x_m - x_n) ds \right\| = \underset{E}{\text{l.u.b.}} \underset{\|f\|=1}{\text{l.u.b.}} \left| \int_E f(x_m(s) - x_n(s)) ds \right|$$

$$\leqq \underset{\|f\|=1}{\text{l.u.b.}} \int_0^1 | f(x_m - x_n) | \, ds$$

$$\leqq ((x_m - x_n))_1;$$

and, on the other hand, if $\text{l.u.b.}_E \|\int_E (x_m - x_n) ds\| < \epsilon$, then for $\|f\| = 1$

$$\int_0^1 | f(x_m - x_n) | \, ds = \left| \int_{E_f^+} f(x_m - x_n) ds \right| + \left| \int_{E_f^-} f(x_m - x_n) ds \right|$$

$$\leqq \left\| \int_{E_f^+} (x_m - x_n) ds \right\| + \left\| \int_{E_f^-} (x_m - x_n) ds \right\| < 2\epsilon.$$

* See [6], p. 368, where this l.u.b. is used to norm the Birkhoff integrable functions.

Thus in $\mathfrak{L}_0^1(\mathfrak{X})$ these two definitions of norm are topologically equivalent; they are not, however, identical.†

In passing we note the obvious facts that if $x(:)$ is in $\mathfrak{L}^{p'}(\mathfrak{X})$, then $x(:)$ is in $\mathfrak{L}^{q'}(\mathfrak{X})$ for $1 \leqq q' \leqq p'$, and that

$$\lim_{m,n\to\infty} ((x_m - x_n))_{p'} = 0 \to \lim_{m,n} ((x_m - x_n))_{q'} = 0 \to \lim_{m,n} ((x_m - x_n))_1 = 0$$

$$\to \lim_{m,n} \underset{E}{\text{l.u.b.}} \left\| \int_E (x_m - x_n) \right\| = 0$$

$$\to \lim_{m,n} \left\| \int_E (x_m - x_n) \right\| = 0$$

for every measurable E, the arrows meaning "implies that."

In the remainder of this section we shall consider pairs p, p' of "conjugate exponents," that is, numbers p, p' connected by the relations $1/p+1/p'=1$ for $1<p<\infty$ (or $1<p'<\infty$), $p'=\infty$ for $p=1$, and $p'=1$ for $p=\infty$, and shall consider the operations from L^p to $\mathfrak{X}$ defined by elements of $\mathfrak{L}_0^{p'}(\mathfrak{X})$. Given L^p we shall deal only with $\mathfrak{L}^{p'}(\mathfrak{X})$, where p and p' are conjugate, and given $\mathfrak{L}^q(\mathfrak{X})$, only with $L^{q'}$, q and q' conjugate.

THEOREM 3.3. *If $x(:)$ is in $\mathfrak{L}^{p'}(\mathfrak{X})$, then for a fixed $\phi(:)$ in L^p the integral*

$$F_\phi(f) = \int_0^1 f(x(s))\phi(s)ds \tag{1}$$

defines a linear functional F_ϕ over $\bar{\mathfrak{X}}$,‡ and the operation

$$F_\phi = U^*(\phi) \tag{2}$$

so defined from L^p to $\bar{\bar{\mathfrak{X}}}$ is linear, with $\|U^*\| = ((x))_{p'}$.

For a fixed $x(:)$ in $\mathfrak{L}^{p'}(\mathfrak{X})$, $(1 \leqq p' \leqq \infty)$, the operation $U(f) = f(x(:))$ from $\bar{\mathfrak{X}}$ to $L^{p'}$ is linear by Theorem 3.2. Then if g_ϕ is the linear functional over $L^{p'}$ generated by the element $\phi(:)$ of L^p, it follows that

$$g_\phi(U(f)) = \int_0^1 f(x(s))\phi(s)ds = F_\phi(f) \tag{3}$$

defines a linear functional $F_\phi(f)$ over $\bar{\mathfrak{X}}$, that is, an element F_ϕ of $\bar{\bar{\mathfrak{X}}}$. If $p' < \infty$, so that g is a linear functional over $L^{p'}$ only if it is so generated, the operation

$$F_\phi = U^*(\phi)$$

† If we take $n^2=2$ in example 4 of [6], then $((x))_1 \geqq 2^{-1/2}$ while $\text{l.u.b.}_E \left\| \int_E x(s)ds \right\| < 2^{-1/2}$.

‡ See [8].

thus defined from $\overline{L^{p'}} = L^p$ to $\overline{\mathfrak{X}}$ is by definition the operation $\overline{U}$ adjoint to U; hence U^* is linear and $\|U^*\| = \|\overline{\overline{U}}\| = \|U\| = ((x))_{p'}$.

If $p' = \infty$, then $\overline{U}$ is from $\overline{L^\infty}$ to $\overline{\mathfrak{X}}$, where $\overline{L^\infty}$ is not an L^p class ([13], p. 875; [16]). But $\overline{L^\infty}$ includes L^1 in the sense that any $\phi(:)$ in L^1 defines a linear functional

$$g_\phi(\phi') = \int_0^1 \phi'(s)\phi(s)ds, \qquad \phi' \text{ in } L^\infty,$$

over L^∞, where $\|g_\phi\|$ is $\int_0^1 |\phi(s)|ds$, the norm of $\phi(:)$ as an element of L^1. Thus $U^*(\phi)$ maps this imbedded subspace onto $\overline{\overline{\mathfrak{X}}}$, and by (2) and equation (3) it is evident that U^* coincides with $\overline{U}$ where U^* is defined. Hence U^* is linear and $\|U^*\| \leqq ((x))_\infty$. If $\|U^*\| < ((x))_\infty$, then there is an f with $\|f\| = 1$ and $\|U^*\| <$ ess. $\sup._S |f(x(s))|$, that is, $\|U^*\| < |f(x(s))|$ on a set of measure greater than zero. Hence there is an f_0 (either $+f$ or $-f$) and a set S_0 such that $\|f_0\| = 1$, $\alpha(S_0) > 0$, and $f_0(x(s)) > \|U^*\| \geqq 0$ on S_0. Letting $\phi_0 = 1/\alpha(S_0)$ times the characteristic function of S_0, we have

$$\|U^*\| = \|U^*\| \cdot \|\phi_0\| \geqq \|U^*(\phi_0)\| = \|F_{\phi_0}\| \geqq |F_{\phi_0}(f_0)|$$
$$= \left| \int_{S_0} f_0(x(s)) \cdot \frac{1}{\alpha(S_0)} ds \right| > \int_{S_0} \|U^*\| \cdot \frac{1}{\alpha(S_0)} ds = \|U^*\|,$$

which is impossible. Thus $\|U^*\| = ((x))_\infty = \|U\| = \|\overline{U}\|$.

THEOREM 3.4. *In order that $x(:)$ be in $\mathfrak{L}_0^{p'}(\mathfrak{X})$, it is necessary and sufficient that $x(:)\phi(:)$ be integrable for every ϕ in L^p. If $x(:)$ is in $L_0^{p'}(\mathfrak{X})$, then the operation*

$$V(\phi) = \int_0^1 x(s)\phi(s)ds \tag{4}$$

from L^p to $\mathfrak{X}$ is linear, with $\|V\| = ((x))_{p'}$. Moreover, the operation $\overline{V}(f)$ adjoint to V is from $\overline{\mathfrak{X}}$ to $L^{p'}$ and assigns to each f in $\overline{\mathfrak{X}}$ the linear functional over L^p generated by the element $U(f) = f(x(:))$ of $L^{p'}$.

Suppose that $x(:)$ is in $\mathfrak{L}_0^{p'}(\mathfrak{X})$. If $\phi(s)$ is a step-function, the integrability of $x(s)\phi(s)$ follows from the linearity and set-additivity of the integral. If $\phi(:)$ in L^p is arbitrary, let $\{\phi_n\}$ be step-functions converging to ϕ in L^p. For any measurable E and any m, n there exists, by B, an f_{mn} in $\overline{\mathfrak{X}}$ such that $\|f_{mn}\| = 1$ and

$$\left\| \int_E x(s)\phi_m(s) - \int_E x(s)\phi_n(s) \right\| = \left| f_{mn}\left(\int_E (\phi_m - \phi_n)x(s)ds \right) \right|;$$

on applying Hölder's inequality, we obtain

$$\left\| \int_E x\phi_m ds - \int_E x\phi_n ds \right\| \leqq \int_E | f_{mn}((\phi_m - \phi_n)x) | ds$$
$$\leqq \| f_{mn}(x(:)) \|_{L^{p'}} \| \phi_m - \phi_n \|_{L^p}$$
$$\leqq ((x))_{p'} \| \phi_m - \phi_n \|_{L^p} \to 0.$$

If $x_E \equiv \lim_n \int_E x(s)\phi_n(s)ds$, then for any f in $\bar{\mathfrak{X}}$ it follows that

$$f(x_E) = \lim_n f\left(\int_E x\phi_n ds \right) = \lim_n \int_E f(x(s)) \cdot \phi_n(s) ds$$
$$= \int_E f(x(s)) \cdot \phi(s) ds = \int_E f(x(s) \cdot \phi(s)) ds,$$

and $x(s) \cdot \phi(s)$ is integrable.

If $x(s)$ is integrable for every ϕ in L^p, then $f(x(s))\phi(s)$ is summable for every ϕ; hence $f(x(:))$ is in $L^{p'}$, that is, $x(:)$ is in $\mathfrak{L}^{p'}(\mathfrak{X})$. The integrability of $x(s)$ results from letting ϕ be the characteristic function of measurable E.

If x_0 in $\mathfrak{X}$ is fixed, the identity

$$F_0(f) \equiv f(x_0), \qquad \text{for all } f \text{ in } \bar{\mathfrak{X}} \tag{5}$$

defines a linear functional F_0 over $\bar{\mathfrak{X}}$. If one sets

$$F_0 = W(x_0),$$

the operation W is linear and norm-preserving from $\mathfrak{X}$ to $\bar{\bar{\mathfrak{X}}}$ and is 1-1 from $\mathfrak{X}$ to its contradomain, the subspace $\bar{\bar{\mathfrak{X}}}'$ of $\bar{\bar{\mathfrak{X}}}$ composed of all elements having a representation of the form (5) ([1], p. 189, proof of Theorem 13). Since W is 1-1 and norm-preserving, and since $\mathfrak{X}$ is a B-space, $\bar{\bar{\mathfrak{X}}}'$ is also a B-space and

$$x = W^{-1}(F)$$

from $\bar{\bar{\mathfrak{X}}}'$ to $\mathfrak{X}$ is linear and norm-preserving.

For ϕ in L^p the integrability of $x(s)\phi(s)$ implies

$$F_\phi(f) = \int_0^1 f(x(s))\phi(s)ds = f\left(\int_0^1 x(s)\phi(s)ds \right),$$

and F_ϕ is in $\bar{\bar{\mathfrak{X}}}'$, with

$$W^{-1}(F_\phi) = \int_0^1 x(s)\phi(s)ds$$

or

$$W^{-1}(U^*(\phi)) = V(\phi).$$

The operation $V(\phi)$ must then be linear, and, since W^{-1} is norm-preserving,

$$\|V\| = \|U^*\| = ((x))_{p'}.$$

To complete the proof suppose f in $\overline{\mathfrak{X}}$, and let f' be the linear functional over L^p defined by $f'(\phi) \equiv f(V(\phi))$; that is, let $f' = \overline{V}(f)$ where $\overline{V}$ is adjoint to V. Then

$$f'(\phi) = f\left(\int_0^1 x(s)\phi(s)ds\right) = \int_0^1 f(x(s)\cdot\phi(s))ds$$

holds for every ϕ in L^p, where $f(x(:))$ is in $L^{p'}$; thus f' is the linear functional over L^p generated by the element $f(x(:)) = U(f)$ of $L^{p'}$.

COROLLARY 3.41. *If $x(s)$ is integrable and $\phi(s)$ is a measurable and essentially bounded real-valued function, then $x(s)\phi(s)$ is integrable.*

4. **Convergence and approximation of integrable functions.** We begin with the following theorem:

THEOREM 4.1. *If $\{x_n(s)\}$ is a sequence of integrable functions converging weakly in measure to $x(s)$, and if $\lim_n \int_E x_n(s)d\alpha$ exists for every measurable E, then $x(s)$ is integrable and $x_E \equiv \lim_n \int_E x_n(s)d\alpha = \int_E x(s)d\alpha$.*

Since $f(x_n(s)) \to f(x(s))$ in measure and since also $f(x_E) = \lim_n f(\int_E x_n(s)d\alpha) = \lim_n \int_E f(x_n(s))d\alpha$, from real-function theory it follows that $f(x(s))$ is integrable and that

$$f(x_E) = \lim_n \int_E f(x_n(s))d\alpha = \int_E f(x(s))d\alpha$$

for any f in $\overline{\mathfrak{X}}$ and any measurable E. Thus $x(s)$ is integrable and $\int_E x(s)d\alpha = x_E = \lim_n \int_E x_n(s)d\alpha$.

THEOREM 4.2. *If $\{x_n\}$ is a Cauchy sequence in $\mathfrak{L}_0^p(\mathfrak{X})$ and $x_n(s) \to x(s)$ weakly in measure, then $x(:)$ is in $\mathfrak{L}_0^p(\mathfrak{X})$ and $((x - x_n))_p \to 0$.*

Since $\{x_n\}$ is a Cauchy sequence,

$$\lim_{m,n\to\infty} \underset{\|f\|=1}{\text{l.u.b.}} (\|f(x_m(:)) - f(x_n(:))\|_{L^p}) = 0;$$

it follows for any f, on considering $f_u = f/\|f\|$, that the functions $f(x_n(:))$ converge as elements in L^p. Since $f(x_n(s)) \to f(x(s))$ in measure, the conclusion is that $f(x(:))$ is in L^p, that is, $x(:)$ is in $\mathfrak{L}^p(\mathfrak{X})$. On the other hand, the hypotheses here imply those of Theorem 4.1, so that $x(:)$ is in $\mathfrak{L}_0^p(\mathfrak{X})$.

Now by hypothesis, given an $\epsilon < 0$ there exists an N_ϵ such that if $m, n \geqq N_\epsilon$, then

$$\underset{\|f\|=1}{\text{l.u.b.}} (\|f(x_m(:)) - f(x_n(:))\|_{L^p}) < \epsilon;$$

thus for any fixed $m \geqq N_\epsilon$ and for any f with $\|f\|=1$ we obtain

$$\|f(x_m(:)) - f(x(:))\|_{L^p} \leqq \epsilon$$

on letting $n\to\infty$ and recalling that $f(x_n(:))\to f(x(:))$ in L^p and that in L^p the norm is a continuous function. This last inequality holds for any $\|f\|=1$ and any $m\geqq N_\epsilon$; then $((x_m-x))_p\leqq\epsilon$ for $m\geqq N_\epsilon$, or $((x_m-x))_p\to 0$.

THEOREM 4.3. *The function $x(s)$ is integrable and measurable if and only if there exist step-functions $\{y_n(s)\}$ such that $\{y_n(s)\}\to x(s)$ a.e. and $((y_n-x))_1\to 0$.*

That the existence of such a sequence implies $x(s)$ measurable and integrable is evident from Theorem 4.2. In the proof of the converse two lemmas will be used.

LEMMA 4.31. *The theorem is true for integrable countably-valued functions.*

Let $x(s)$ have the constant value x_i on E_i, the sets $\{E_i\}$ being countable, measurable, and disjoint, and let $S=\sum E_i$. Since $\sum_i \int_{E_i} x(s)d\alpha$ converges unconditionally, there exists (Theorem 2.32 and Lemma 2.31) for each $1/2^j$ an N_j such that $\|f\|\leqq 1$ implies

$$\sum_{N_j+1}^{\infty} | f(x_i) | \alpha(E_i) = \sum_{N_j+1}^{\infty} \left| f\left(\int_{E_i} x(s)d\alpha\right)\right| < 1/2^j,$$

with $N_1<N_2<\cdots$. By defining $y_n(s)=x_i$ on E_i for $i\leqq N_n$ and $y_n(s)=\theta$ elsewhere, one obtains $\{y_n(s)\}\to x(s)$ everywhere and

$$\int_S | f(x(s) - y_n(s)) | d\alpha = \sum_{N_n+1}^{\infty} \int_{E_i} | f(x(s)) | d\alpha < 1/2^n$$

for $\|f\|\leqq 1$; hence $((x-y_n))_1\to 0$.

LEMMA 4.32. *If $x(s)$ is measurable and integrable, it can be uniformly approximated, except possibly on a set of measure zero, by integrable countably-valued functions.*

In virtue of Corollary 1.12 there are countably-valued functions $x_n(s)$ such that $\|x(s)-x_n(s)\|<1/n$ uniformly for s in S_0, where $\alpha(S_0)=\alpha(S)$. The measurable function $x-x_n$ is then essentially bounded, therefore Bochner integrable, and hence $(\mathfrak{X})$ integrable, which implies integrability for $x_n=x-(x-x_n)$.

Returning to the proof of the theorem, we suppose $x(s)$ integrable and measurable and $x_n(s)$ a countably-valued integrable function within $1/n$ of $x(s)$ essentially uniformly (Lemma 4.32). Then for $\|f\|\leqq 1$ we have

$$\int_S | f(x - x_n) | d\alpha \leqq \int_S \|x - x_n\| d\alpha < \alpha(S)\cdot 1/n,$$

so that $((x-x_n))_1<\alpha(S)\cdot 1/n$.

If n is fixed, the integrable c.v. function $x_n(s)$ is a.e. the limit of a sequence of step-functions, which therefore converge almost uniformly ([5], p. 442) to $x_n(s)$. Hence there exists a step-function $y_n(s)$ such that $\|x_n(s)-y_n(s)\|<1/n$ except on a set of measure at most $1/2^n$, while (from the second conclusion of the first lemma)

$$((x_n - y_n))_1 < \alpha(S)\cdot 1/n.$$

Collecting inequalities, we have $\|x(s)-y_n(s)\|<2/n$ except on a set of measure at most $1/2^n$, and $((x-y_n))_1<\alpha(S)\cdot 2/n$. It follows that $y_n(s)\to x(s)$ almost uniformly, and therefore a.e., and that $((x-y_n))_1\to 0$.

COROLLARY 4.33. *If $\mathfrak{X}$ is separable, then $\mathfrak{L}_0^1(\mathfrak{X})$ is separable.*

5. The ($\mathfrak{X}$) integral as related to others. In addition to the Bochner (Bn), Birkhoff (Bk), and Gelfand extensions of the Lebesgue integral, and Dunford's first integral ([5]; this is equivalent to Bochner's, cf. [9], p. 475), two other definitions, also due to Dunford, have been given ([7], [8]). The last two integrals will be here denoted as the (D1) integral and the (D2) integral, respectively. The first, a direct generalization of the Bochner integral, is defined as follows: $x(s)$ is (D1) integrable if and only if there exist step-functions $x_n(s)$ such that $x_n(s)\to x(s)$ a.e. and $\lim_n \int_E x_n(s)d\alpha$ exists for every measurable E. This limit is (D1) $\int_E x(s)d\alpha$. The second, as we have noted earlier, is a generalization of the ($\mathfrak{X}$) integral; it requires merely that $x(:)$ be in $\mathfrak{L}^1(\mathfrak{X})$ and defines (D2) $\int_E x(s)d\alpha$, to be the element F_E of $\overline{\overline{\mathfrak{X}}}$ given by $F_E(f)=\int_E f(x(s))d\alpha$.

In paragraph 2.21 we noted that the ($\mathfrak{X}$) integral includes both the (Bn) and the (Bk). The connection between the (D1) and (D2) integrals and the ($\mathfrak{X}$) integral will now be considered.

THEOREM 5.1. *A function $x(s)$ is measurable and integrable if and only if $x(s)$ is* (D1) *integrable. The integral of $x(s)$ has the same value for the two definitions.*

This theorem is seen to follow immediately from Theorem 4.3 and Theorem 4.1.

COROLLARY 5.11. *For measurable functions the* ($\mathfrak{X}$), *(Bk), and (D1) integrals are equivalent.*

In view of the theorem and the fact that (Bk) integrability implies ($\mathfrak{X}$) integrability to the same value, it is sufficient to show that a measurable integrable $x(s)$ is (Bk) integrable; that is, ([6], Theorem 13), given $\epsilon>0$ there exist disjoint measurable sets $\{E_i\}, i=0, 1, \cdots, n, \cdots$, such that $\sum_{i=0}^{\infty} E_i=S$, the series $\sum_{i=0}^{\infty}\alpha(E_i)x(s_i)$ is unconditionally convergent for any choice of s_i in

E_i, and s_i, s_i' in E_i implies $\|\sum_1^\infty \alpha(E_i)x(s_i) - \sum_1^\infty \alpha(E_i)x(s_i')\| < \epsilon$. From Theorem 4.32 there exist a c.v. integrable $y(s)$ and a set E_0 such that $\alpha(E_0) = 0$

$$\|x(s) - y(s)\| < \epsilon/4\alpha(S), \qquad \text{for } s \text{ in } S_0 = S - E_0, \tag{1}$$

and

$$\left\| \int_{S_0} x(s)d\alpha - \int_{S_0} y(s)d\alpha \right\| < \epsilon/4. \tag{2}$$

Let $\{E_i\}$, $(i = 1, 2, \cdots)$, be disjoint measurable sets with $\sum_1^\infty E_i = S_0$ and $y(s) = y_i$ on E_i. Then $\int_{S_0} y(s)d\alpha = \sum_1^\infty \alpha(E_i)y_i$ is unconditionally convergent; hence, since (1) implies that for arbitrary s_i in E_i, $(i = 0, 1, \cdots)$, the inequality $\sum_1^\infty \|\alpha(E_i)(x(s_i) - y_i)\| < \epsilon/4$ holds, the series

$$\sum_0^\infty \alpha(E_i)x(s_i) = \sum_1^\infty \alpha(E_i)x(s_i) = \sum_1^\infty \alpha(E_i)(x(s_i) - y_i) + \sum_1^\infty \alpha(E_i)y_i$$

is unconditionally convergent, being the sum of two such series. Moreover, $\|\sum_0^\infty \alpha(E_i)x(s_i) - \sum_1^\infty \alpha(E_i)y_i\| < \epsilon/4$, and this fact combined with (2) gives $\|\sum_0^\infty \alpha(E_i)x(s_i) - \int_{S_0} x(s)d\alpha\| < \epsilon/2$, which completes the proof.

A consequence of Corollary 5.11 is the failure of the Fubini theorem to hold for the (D1), ($\mathfrak{X}$), and (D2) integrals, since ([6], example 6) the theorem fails for the (Bk) integral when $\mathfrak{X}$ is the separable space L^2.

THEOREM 5.2. *The integrals of two measurable and integrable functions $x(s)$, $y(s)$ coincide over every measurable set if and only if $x(s) = y(s)$ a.e.*

If the integrals of measurable $x(s)$ and $y(s)$ coincide everywhere, then $z(s) = x(s) - y(s)$ is separably-valued and has its integral vanishing identically. Let S_0 be a subset in S such that $\alpha(S - S_0) = 0$ and $z(S_0) \subset \mathfrak{Y}$, where $\mathfrak{Y}$ is a separable subspace of $\mathfrak{X}$. Letting $\{g_i\}$ be a sequence weakly dense in the surface of the unit sphere of $\bar{\mathfrak{Y}}$, we find as in paragraph 3.11 that $A = \sum A_i$, where $A = S_0[\|z(s)\| > 0]$ and $A_i = S_0[|g_i(z(s))| > 0]$. Since $\int_E z(s)d\alpha = \theta$ for all E, it results that $\int_E g_i(z(s))d\alpha \equiv 0$ in E for each i, and therefore that $|g_i(z(s))| = 0$ a.e. in S_0. Thus $\alpha(A_i) = 0$ which implies $\alpha(A) = 0$, so that $z(s) = \theta$ a.e. in S_0 and hence a.e. in S.*

The remaining part of the theorem follows from (4) of paragraph 2.11.

THEOREM 5.3. *If measurable $x(s)$ is in $\mathfrak{L}^1(\mathfrak{X})$, then $x(s)$ is integrable if and only if $F_E = (\mathrm{D2})\int_E x(s)d\alpha$ is absolutely continuous.*

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* Theorem 5.2 can be regarded, in the light of 5.11, as a corollary of Theorem 24 of [6]. In the proofs of Corollary 5.11 and Theorem 5.2 we have essentially repeated Birkhoff's arguments for Theorems 22 and 24 of [6].

If $x(s)$ is in $\mathfrak{L}_0^1(\mathfrak{X})$, then

$$F_E(f) = \int_E f(x(s))d\alpha = f\left(\int_E x(s)d\alpha\right)$$

for all f; therefore $\|F_E\| = \|\int_E x(s)d\alpha\|$ where the first norm is taken over $\bar{\bar{\mathfrak{X}}}$ and the second over $\mathfrak{X}$. The absolute continuity of F_E now follows from Theorem 2.5.

Suppose $x(s)$ measurable and in $\mathfrak{L}^1(\mathfrak{X})$. Let $\mathfrak{Y}$ be a separable subspace containing almost all of the values $x(s)$; then $x(:)$ is in $\mathfrak{L}^1(\mathfrak{Y})$, since any linear functional over $\mathfrak{Y}$ is extensible to $\mathfrak{X}$. To show that $x(:)$ is in $\mathfrak{L}_0^1(\mathfrak{Y})$, and therefore that $x(:)$ is in $\mathfrak{L}_0^1(\mathfrak{X})$, it is sufficient, $\mathfrak{Y}$ being separable, to prove that for any measurable E the linear functional $F_E(g)$ over $\bar{\mathfrak{Y}}$ is weakly continuous ([1], p. 131). If $g_n(y) \rightarrow g_0(y)$ for every y in $\mathfrak{Y}$, then $\|g_n\| \leqq K$, $(n = 0, 1, 2, \cdots)$; and if F_E is a.c., then

$$\left| \int_E g_n(x(s))d\alpha \right| = | F_E(g_n) | \leqq \|g_n\| \cdot \|F_E\| \leqq K\epsilon$$

for $\alpha(E) < \delta$. The integrals $\int_E g_n(x(s))d\alpha$ are thus equi-absolutely continuous; since $g_n(x(s)) \rightarrow g_0(x(s))$ for almost all s, it follows that

$$F_E(g_n) = \int_E g_n(x(s))d\alpha \rightarrow \int_E g_0(x(s))d\alpha = F_E(g_0)$$

for every E. Thus for any fixed E the functional $F_E(g)$ over $\bar{\mathfrak{Y}}$ is weakly continuous, so that a y_E in $\mathfrak{Y}$ exists such that $g(y_E) = F_E(g)$ for all g in $\bar{\mathfrak{Y}}$. Since any linear functional over $\mathfrak{X}$ defines one over $\mathfrak{Y}$, and since y_E is in $\mathfrak{Y}$, it follows that $F_E(f) = f(y_E)$ for all f in $\bar{\mathfrak{X}}$, and $x(s)$ is integrable.

Corollary 5.31. *If measurable $x(s)$ is in $\mathfrak{L}^p(\mathfrak{X})$, $(p > 1)$, then $x(:)$ is in $\mathfrak{L}_0^p(\mathfrak{X})$. If $\mathfrak{X}$ is weakly complete, this is also true for $p = 1$.*

Suppose $p > 1$. If

$$F_E(f) \equiv \int_E f(x(s))d\alpha = \int_0^1 f(x(s))\phi_E(s)ds,$$

where $\phi_E(s)$ is the characteristic function of E, then

$$| F_E(f) | \leqq \|f(x(:))\|_{L^p} \cdot \|\phi_E(:)\|_{L^{p'}} \leqq ((x))_p \cdot | \alpha(E) |^{1/p'} < \epsilon$$

if $\|f\| \leqq 1$ and $\alpha(E) < \epsilon^{p'}/[((x))_p]^{p'}$. Thus $\|F_E\| \leqq \epsilon$ when $\alpha(E)$ is sufficiently small; hence $x(:)$ is integrable by the theorem.

Suppose $\mathfrak{X}$ weakly complete and $p = 1$. From Corollary 1.12 and Theorem 4.2 it clearly suffices to consider c.v. functions. For such an $x(s)$

$$\infty > \int_E |f(x(s))| \, d\alpha = \sum |f(x_i)| \cdot \alpha(E_i \cdot E),$$

so that $\sum x_i \alpha(E_i \cdot E)$ is a weakly convergent series in the weakly complete space $\mathfrak{X}$. Then there exists an x_E such that

$$f(x_E) = \sum f(x_i) \cdot \alpha(E_i \cdot E) = \int_E f(x(s)) d\alpha,$$

and $x(s)$ is integrable.

COROLLARY 5.32. *If* $1 \leqq p \leqq \infty$ *and* $1 \leqq q < \infty$, *then* $\mathfrak{L}^p(L^q) = \mathfrak{L}_0^p(L^q)$ *and* $\mathfrak{L}^p(l^q) = \mathfrak{L}_0^p(l^q)$.

6. **Concerning completely continuous operations.** We prove the following lemma:

LEMMA 6.11. *If $x(s)$ is a step-function from* $[0, 1]$ *to $\mathfrak{X}$, the operations*

$$U(f) = f(x(:))$$

$$V(\phi) = \int_0^1 x(s)\phi(s) ds$$

from $\overline{\mathfrak{X}}$ to $L^{p'}$ and from L^p to $\mathfrak{X}$ are c.c. (completely continuous).

If U is c.c., so are $\overline{U}$ and (Theorem 3.3) U^*; since $V(\phi) = W^{-1}(U^*(\phi))$, where W^{-1} is linear, this implies the same property for V. Suppose $x(s) = x_i$ on E_i, $(\sum_1^N E_i = [0, 1])$; then $U(f)$ is the function having the constant value $f(x_i)$ on E_i. If $\|f_n\| \leqq K$, so that $|f_n(x_i)| \leqq K \cdot \|x_i\|$, then clearly there exists a subsequence $\{f_n'\}$ such that the functions $\{f_n'(x(s))\}$ converge uniformly over $[0, 1]$ and hence $\{f_n'(x(:))\}$ is a convergent sequence in $L^{p'}$ for any p'.

LEMMA 6.12. *If $T(\phi)$ is linear from L^p, $(p < \infty)$, to a subspace $\mathfrak{M}$ of finite dimension in $\mathfrak{X}$, then there exists an $x(:)$ in $\mathfrak{L}_0^{p'}(\mathfrak{X})$ defining $T(\phi)$, that is,*

$$T(\phi) = \int_0^1 x(s)\phi(s) ds$$

for all ϕ in L^p; and $x(:)$ is the limit in $\mathfrak{L}_0^{p'}(\mathfrak{X})$ of step-function elements.

Let x_i, f_i, $(i = 1, 2, \cdots, n)$, be a complete biorthogonal system in $\mathfrak{M}$, so that

$$T(\phi) = \sum_1^n g_i(\phi) \cdot x_i,$$

where $g_i(\phi) = f_i(T(\phi))$. The functional f_i being extensible from $\mathfrak{M}$ to $\mathfrak{X}$, the last equation implies that $g_i(\phi)$ is a linear functional over L^p; since $p<\infty$, there is an element $\bar{\phi}_i$ in $L^{p'}$ generating $g_i(\phi)$. Set

$$(1) \qquad x(s) = \sum_1^n \bar{\phi}_i(s)\cdot x_i.$$

Obviously $x(:)$ is in $\mathfrak{L}_0^{p'}(\mathfrak{X})$; and

$$\int_0^1 x(s)\phi(s)ds = \sum_1^n x_i\cdot \int_0^1 \bar{\phi}_i(s)\phi(s)ds$$
$$= \sum x_i g_i(\phi) = T(\phi).$$

The remainder of the lemma results from approximating, in $L^{p'}$, each $\bar{\phi}_i$ in (1) by a real-valued step-function.

THEOREM 6.1. *A necessary and sufficient condition that a linear operation $V(\phi)$ from L^p, $(1<p<\infty)$, to $\mathfrak{X}$ be c.c. is that it be the strong limit of a sequence of operations defined by step-functions.*

If $V(\phi)$ from L^p to $\mathfrak{X}$ is c.c. and $1<p<\infty$, then ([14], p. 197) V is the limit of operations $\{T_n\}$ where each T_n maps L^p into a finite-dimensional subspace of $\mathfrak{X}$. By Lemma 6.12 there exists a step-function $x_n(:)$ whose corresponding operation V_n is in norm within $1/n$ of T_n, so that $\|V-V_n\|\to 0$.

The converse results from Lemma 6.11 and the fact that the limit under the norm of a sequence of c.c. operations is again c.c.

THEOREM 6.2. *If $x(s)$ is measuralbe and integrable, then $U(f)=f(x(:))$ from $\mathfrak{X}$ to L^1 and $V(\phi)=\int_0^1 x(s)\phi(s)ds$ from L^∞ to $\mathfrak{X}$ are c.c.*

This follows from Theorem 4.3, and Lemma 6.11, and the fact that for $x(:)$ in $\mathfrak{L}_0^1(\mathfrak{X})$

$$\|U\| = \|V\| = ((x))_1.$$

7. **Representation of integrals by real-valued kernels.** We make the following definition:

DEFINITION 7.1. $L^{p'q}$, $(1\leqq p'\leqq\infty,\ 1\leqq q\leqq\infty)$, *is the class of real-valued functions $F(s, t)$ defined on the unit square and having the following properties:*

(a) *$F(s, t)$ is measurable in (s, t).*

(b) *For each s the function $F(s, :)$ is in L^q.*

(c) *The iterated integral*

$$(1) \qquad \int_0^1 \left\{ \int_0^1 F(s,t)\bar{\psi}(t)dt \right\} \phi(s)ds$$

exists for every $\bar{\psi}$ in $L^{q'}$ and ϕ in L^p. For $p'=\infty$ the restriction

$$K_F \equiv \underset{\|\bar{\psi}\|_{L^{q'}}=1}{\text{l.u.b.}} \left(\underset{S}{\text{ess. sup.}} \left| \int_0^1 F(s, t)\bar{\psi}(t)dt \right| \right) < \infty$$

is imposed.

LEMMA 7.21. *If $x(s)$ from $[0, 1]$ to L^q, $(1 \leqq q \leqq \infty)$, is weakly measurable, then there exists a function $F(s, t)$ from the unit square to the reals that is measurable in (s, t) and such that $x(s) = F(s, :)$ as elements in L^q for each s in $[0, 1]$. Any two such "measurable representations" of $x(s)$ differ on at most a set of plane measure zero.*

For $q < \infty$ this is an immediate consequence of Corollary 1.11 and a theorem of Dunford ([9], Theorem 3.1). The case $q = \infty$ results from the preceding, since $x(s)$, if weakly measurable to L^∞, is also weakly measurable when considered as defined to L^1.

THEOREM 7.2. *If $x(:)$ is in $\mathfrak{L}^{p'}(L^q)$, then any measurable representation $F(s, t)$ of $x(s)$ is in $L^{p'q}$. If $q < \infty$ and $F(s, t)$ is in $L^{p'q}$, then the abstract function $x(s) \equiv F(s, :)$ is in $\mathfrak{L}^{p'}(L^q) = \mathfrak{L}_0^{p'}(L^q)$.*

Since any $\bar{\psi}$ in $L^{q'}$ defines a functional $f_{\bar{\psi}}$ over L^q, $x(:)$ in $\mathfrak{L}^{p'}(L^q)$ implies that

$$f_{\bar{\psi}}(x(s)) = \int_0^1 F(s, t)\bar{\psi}(t)dt$$

is in $L^{p'}$ and hence that (c) of Definition 7.1 is satisfied; since $F(s, t)$ is a measurable representation of $x(s)$, the remaining two conditions are also fulfilled.

Conversely, if $F(s, t)$ is in $L^{p'q}$, then $x(s) \equiv F(s, :)$ is defined from $[0, 1]$ to L^q; and if $q < \infty$, so that every linear functional $f(\psi)$ over L^q is generated by an element $\bar{\psi}$ of $L^{q'}$, it follows that $f(x(s)) = \int_0^1 F(s, t)\bar{\psi}(t)dt$ is in $L^{p'}$; for if the iterated integral in (c) of Definition 7.1 exists for all ϕ in L^p, then the right-hand side of the above equation is in $L^{p'}$ ([1], p. 85). Thus $x(:)$ is in $\mathfrak{L}^{p'}(L^q) = \mathfrak{L}_0^{p'}(L^q)$.

THEOREM 7.3. *If $F(s, t)$ is a measurable representation of an element $x(:)$ of $\mathfrak{L}_0^{p'}(L^q)$, then a necessary and sufficient condition that $V(\phi) = \int_0^1 x(s)\phi(s)ds$ from L^p to L^q be expressible as*

$$V(\phi) = \int_0^1 F(s, t)\phi(s)ds \tag{2}$$

is that ϕ in L^p and $\bar{\psi}$ in $L^{q'}$ imply that

$$\int_0^1 \bar{\psi}(t) \left\{ \int_0^1 F(s, t)\phi(s)ds \right\} dt = \int_0^1 \phi(s) \left\{ \int_0^1 F(s, t)\bar{\psi}(t)dt \right\} ds. \tag{3}$$

From Theorem 7.2, $F(s, t)$ is in $L^{p'q}$, so that the right-hand integral in (3) exists. The elements $\bar{\psi}$ of $L^{q'}$ define linear functionals $f_{\bar{\psi}}$ over L^q; hence if (2) is true, then

$$f_{\bar{\psi}}\left(\int_0^1 F(s, t)\phi(s)ds\right) = f_{\bar{\psi}}\left(\int_0^1 x(s)\phi(s)ds\right),$$

or

$$\int_0^1 \bar{\psi}(t)\left\{\int_0^1 F(s, t)\phi(s)ds\right\}dt = \int_0^1 \phi(s)\cdot f_{\bar{\psi}}(F(s, :))ds$$
$$= \int_0^1 \phi(s)\cdot\left\{\int_0^1 F(s, t)\bar{\psi}(t)dt\right\}ds.$$

On the other hand suppose that $x(:)$ is in $\mathfrak{L}_0{}^{p'}(L^q)$, so that $x(:)$ defines the operation $V(\phi) = \int_0^1 x(s)\phi(s)ds$ and suppose that $x(s)$ has a measurable representation $F(s, t)$ satisfying (3). By Theorem 3.4 the operation $\bar{V}$ adjoint to V assigns to each f in $\overline{L^q}$ the linear functional over L^p generated by the element $\bar{\phi} = f(x(:))$ of $L^{p'}$.

Let ϕ be any element of L^p, and let $\mathfrak{F}$ be the set of functionals $f_{\bar{\psi}}$ over L^q generated by elements $\bar{\psi}$ of $L^{q'}$. Then for all $f_{\bar{\phi}} = \bar{V}(f_{\bar{\psi}})$ with $f_{\bar{\psi}}$ in $\mathfrak{F}$

$$f_{\bar{\phi}}(\phi) = \int_0^1 \bar{\phi}(s)\phi(s)ds = \int_0^1 f_{\bar{\psi}}(x(s))\phi(s)ds$$
$$= \int_0^1 \phi(s)\left\{\int_0^1 F(s, t)\bar{\psi}(t)dt\right\}ds,$$

or, if (3) is applied,

$$f_{\bar{\phi}}(\phi) = \int_0^1 \bar{\psi}(t)\cdot\left\{\int_0^1 F(s, t)\phi(s)ds\right\}dt,$$

where $\int_0^1 F(s, t)\phi(s)ds$ must be an element of L^q. Then for a fixed ϕ in L^p,

$$f_{\bar{\phi}}(\phi) = f_{\bar{\psi}}\left(\int_0^1 F(s, t)\phi(s)ds\right)$$

whenever $f_{\bar{\psi}}$ is in $\mathfrak{F}$ and $f_{\bar{\phi}} = \bar{V}(f_{\bar{\psi}})$. But by definition of $\bar{V}$ the equation $f_{\bar{\phi}}(\phi) = f_{\bar{\psi}}(V(\phi))$ holds for all ϕ in L^p; hence, for any fixed ϕ,

$$f_{\bar{\psi}}(V(\phi)) = f_{\bar{\psi}}\left(\int_0^1 F(s, t)\phi(s)ds\right)$$

for all $f_{\bar{\psi}}$ in $\mathfrak{F}$. This implies (2), since the set $\mathfrak{F}$ is a total set of functionals over L^q.

COROLLARY 7.31. *If* $x(:)$ *in* $\mathfrak{L}_0^{p'}(L^q)$ *has a measurable representation* $F(s, t)$ *such that* $\int_0^1\phi(s)\{\int_0^1|F(s, t)|\bar{\psi}(t)dt\}ds$ *exists for every* ϕ *in* L^p *and* $\bar{\psi}$ *in* $L^{q'}$, *then*

$$\int_0^1 x(s)\phi(s)ds = \int_0^1 F(s, t)\phi(s)ds$$

for all ϕ *in* L^p.

This follows from Theorem 7.3 and a well known theorem of Tonelli ([2], p. 75).

COROLLARY 7.32. *If* $F(s, t)$ *is in* $L^{\infty q}$, $(q < \infty)$, *then*

$$\operatorname{ess.\,sup.}_S \left(\int_0^1 |F(s, t)|^q dt\right)^{1/q} = K_F < \infty,$$

and

$$V(\phi) = \int_0^1 F(s, t)\phi(s)ds$$

is defined and linear from L^1 *to* L^q.

By Theorem 7.2, $x(s) \equiv F(s, :)$ is in $\mathfrak{L}^\infty(L^q) = \mathfrak{L}_0^\infty(L^q) = S_{\infty q}$; hence (Definition 3.1)

$$\infty > K_F = \operatorname{ess.\,sup.}_S \|x(s)\| = \operatorname{ess.\,sup.}_S \left(\int_0^1 |F(s, t)|^q dt\right)^{1/q}.$$

Clearly $x(:)$ satisfies the hypotheses of Corollary 7.31; hence the operation $V(\phi) = \int_0^1 x(s)\phi(s)ds$ can be written as $V(\phi) = \int_0^1 F(s, t)\phi(s)ds$.

THEOREM 7.4. *If* $q < \infty$, *if* $F(s, t)$, *defined and measurable from the unit square to the reals, has* $F(s, :)$ *in* L^q *for almost all* s, *and if*

$$U(\bar{\psi}) = \int_0^1 F(s, t)\bar{\psi}(t)dt$$

is defined from $L^{q'}$ *to* L^1, *then* U *is linear and c.c.*

If in addition $V(\phi) = \int_0^1 F(s, t)\phi(s)ds$* *is defined from* L^∞ *to* L^q *and* $\int_0^1 V(\phi)\cdot\psi dt = \int_0^1 U(\bar{\psi})\phi ds$ *for every pair* $\bar{\psi}$ *in* $L^{q'}$, ϕ *in* L^∞, *then* $V(\phi)$ *is also linear and c.c.*

This results from Definition 7.1, Theorem 7.2, Corollary 5.32, Theorem 7.3, and Theorem 6.2.

8. **Concerning differentiation.** In this section S is taken to be euclidean, and $\alpha(E)$ is the Lebesgue measure function.

* See [9], Theorem 6.1, where for a more restricted type of $F(s, t)$ this operation is shown to be defined and c.c. from L^p to L^q, $(1 < p \leq \infty, 1 \leq q < \infty)$.

The relation between the Bochner integral and differentiation is that an additive a.c. BV function of figures is a.e. strongly differentiable if and only if it is a Bochner integral; and the derivative, when it exists a.e., is (Bn) integrable to the original function ([18], p. 410, footnote). However ([6], example 1), an ($\bar{\mathfrak{X}}$) integral may be nowhere weakly differentiable (paragraph 1.14); there is therefore no hope for a theorem similar to the above and dealing with the ($\mathfrak{X}$) integral and either strong or weak derivatives. But if another and more general definition of "derivative" is employed, the corresponding theorem can be stated.

A function $X(R)$ defined to $\mathfrak{X}$ from the figures in a rectangle R_0 of euclidean n-space will be said to be *pseudo-differentiable* or to have a *pseudo-derivative* if there exists an $x(s)$ from R_0 to $\mathfrak{X}$ such that for every f in $\bar{\mathfrak{X}}$ the figure function $f(X(R))$ is differentiable a.e. to the value $f(x(s))$. The function $x(s)$ is a *pseudo-derivative of* $X(R)$.

A necessary and sufficient condition that an additive a.c. function $X(E)$ *of measurable sets in* R_0 *be pseudo-differentiable is that the function* $X(E)$ *be an* ($\mathfrak{X}$) *integral.*

The sufficiency is obvious; an ($\mathfrak{X}$) integral has the integrand as a pseudo-derivative.

If $X(E)$ has a pseudo-derivative $x(s)$, then if $X(E)$ is additive and a.c., so is $f(X(E))$; hence

$$f(X(E)) = \int_E \left(\frac{d}{ds} f(X(E))\right) ds = \int_E f(x(s))ds,$$

so that $x(s)$ is integrable to the value $X(E)$.

If $\mathfrak{X}$ is weakly complete, the preceding theorem is true for additive a.c. functions of figures. For if $X(R)$ has $x(s)$ for a pseudo-derivative, we have $f(X(R)) = \int_R f(x(s))ds$; if Q is any open set, then $Q = \sum I_n$, where the $\{I_n\}$ are disjoint intervals, and thus

$$\sum | f(X(I_n)) | = \sum \left| \int_{I_n} f(x(s))ds \right| \leqq \int_Q | f(x(s)) | \, ds < \infty .$$

Since $\mathfrak{X}$ is weakly complete, it follows that $\sum X(I_n)$ has property (O) and therefore that $\sum X(I_n)$ is convergent. Designating this sum by $X(Q)$ we have

$$f(X(Q)) = \sum f(X(I_n)) = \int_Q f(x(s))ds.$$

Finally, if E is an arbitrary measurable set, and if $\{Q_n\}$ are open sets containing E with $\alpha(Q_n) \to \alpha(E)$, then

$$f(X(Q_n)) = \int_{Q_n} f(x(s))ds \to \int_E f(x(s))ds;$$

and, again using the weak completeness of $\mathfrak{X}$, we find that $\{X(Q_n)\}$ converges weakly to an element $X(E)$, and $f(X(E)) = \int_E f(x(s))ds$. Hence $x(s)$ is ($\mathfrak{X}$) integrable to the set-function $X(E)$.

The pseudo-derivative suggests a corresponding "descriptive" definition of an integral: a function $x(s)$ is "integrable" if and only if there exists an additive a.c. function $X(E)$ of measurable sets which has $x(s)$ for a pseudo-derivative; the integral of $x(s)$ over E is $X(E)$. From the above theorem this descriptive integral is the ($\mathfrak{X}$) integral.

If $X(E)$ has a.e. a weak derivative $x(s)$, then $x(s)$ is a pseudo-derivative; hence if $X(E)$ is additive a.c. and weakly differentiable a.e., this weak derivative is ($\mathfrak{X}$) integrable to the value $X(E)$; moreover, by Theorem 1.2 the derivative is measurable, so that $X(E)$ is the ($\mathfrak{X}$) integral of a measurable integrable function, its weak derivative. We have been unable to resolve the converse question: does a measurable integrable function have its integral weakly differentiable a.e.? If so, then by Theorem 5.2 this weak derivative coincides a.e. with the original function.

In conclusion we recall that if $\mathfrak{X}$ is separable and $x(s)$ is essentially bounded and weakly measurable, then $x(s)$ is (Bn) integrable, and $\int_E x(s)ds = (\text{Bn})\int_E x(s)ds$ is therefore strongly differentiable a.e. to $x(s)$. In particular if $\bar{\mathfrak{X}}$ is separable and $x(s)$ is (Bk) integrable and essentially bounded, then $(\text{Bk})\int_E x(s)ds = \int_E x(s)ds$ is strongly differentiable a.e. to $x(s)$. This gives an affirmative answer to a question raised in [6] (p. 378).

9. **Examples.** We give the following examples:

9.1. Let $\mathfrak{X}$ be the space of real functions $\phi(t)$ bounded on $[0, 1]$, with $\|\phi\| = \text{l.u.b.}_t |\phi(t)|$. Let E be a non-measurable set in $S = [0, 1]$; define $x(s) = \theta$ for s in $S - E$, and set $x(s)$ equal to the characteristic function of the point s for s in E. Then $x(s)$ is Graves integrable (see [3]) over every measurable set E to the value θ; hence $x(s)$ is (Bk) integrable and integrable to the same value. The function $x(s)$ is weakly measurable and integrable, yet $\|x(s)\|$ is not measurable. And the set function $X(E) = \int_E x(s)ds \equiv \theta$ is weakly differentiable everywhere to the value θ and thus has two pseudo-derivatives, $x(s)$ and the function identically θ, that differ on a set of positive measure.

9.2. Let $S = [0, 1]$, let $\alpha(E) = mE$, where m is the Lebesgue measure function, and let $\mathfrak{X} = c$, the separable space of convergent sequences of reals. On $J_n = (1/2^n, 1/2^{n-1}]$ define $x(s) = 2^n x_n$, where x_n is the nth unit vector in c.

The function $x(s)$ is in $\mathfrak{L}^1(c)$, that is, $x(s)$ is (D2) integrable, and if $E_n = E \cdot J_n$, then

$$\text{(D2)} \int_E x(s)ds = F_E = \{2^1 mE_1, \cdots, 2^n mE_n, \cdots\},$$

an element of $\bar{\bar{c}} = l^\infty$, where l^∞ is the space of bounded sequences. Here F_E is not a.c., so that, from Theorem 5.3, $x(:)$ is not in $\mathfrak{L}_0^1(c)$;* hence in Corollary 5.31 the hypothesis of weak completeness cannot be omitted. Nor is the set function F_E c.a., since the series $\sum_n F_{J_n}$ in $\bar{\bar{c}}$ is not even weakly convergent to an element; to prove this it is sufficient to consider the linear functional over $l^\infty = \bar{\bar{c}}$ defined by Hildebrandt's generalized Stieltjes integral ([13], p. 870) using Banach's measure function ([1], p. 231).

If $y(s) = x(s)$ on $(0, 1]$, if $y(s) = -x(-s)$ for s in $[-1, 0)$, and if $y(0) = \theta$, then $y(s)$ is integrable over $[-1, 1]$ in the sense of Definition 2.1, yet $y(s)$ is not $(\mathfrak{X})$ integrable; this is in contrast to the (Bk) definition, since (Bk) integrability over S implies (Bk) integrability over every measurable subset of S ([6], Theorem 14).

9.3 Let $\{x_n\}$ be a complete orthonormal sequence in L^2, and define $x(s) = 2^n x_n$ on $J_n = (1/2^n, 1/2^n + 1/2^{2n}]$ and $x(s) = \theta$ elsewhere in $[0, 1]$. Then $x(s)$ is (Bn) integrable, and $x(:)$ is in $\mathfrak{L}_0^2(L^2)$, while $\int_0^1 \|x(s)\|^2 ds = \infty$.

This example shows that in Theorem 6.2 we cannot replace $\mathfrak{L}_0^1(\mathfrak{X})$ by $\mathfrak{L}_0^{p'}(\mathfrak{X})$, $(1 \leqq p' < \infty)$. For in $\mathfrak{L}_0^2(L^2)$ there are no step-function elements within $1/2$ of $x(:)$, so that $V(\phi) = \int_0^1 x(s)\phi(s)ds$ from L^2 to L^2 is not c.c. Suppose that $0 < \epsilon < 1/2$ and that $y(s)$ is a step-function taking the values $y_1, \cdots, y_k$. If f_n is the linear functional over L^2 generated by x_n, then $\sum_{n=1}^\infty |f_n(y_i)|^2 < \infty$ for each i; hence there exists an N such that

$$|f_N(y_i)| < \epsilon, \qquad i = 1, \cdots, k.$$

If s is in J_N, then

$$\begin{aligned} |f_N(x(s) - y(s))| &= |f_N(2^N x_N - y_i)| \\ &= 2^N \cdot \left| 1 - \frac{1}{2^N} f_N(y_i) \right| > 2^N \left(1 - \frac{\epsilon}{2^N} \right); \end{aligned}$$

whence

$$\begin{aligned} ((x - y))_2 &\geqq \left(\int_{J_N} |f_N(x - y)|^2 ds \right)^{1/2} \\ &> \left(\frac{1}{2^{2N}} \cdot 2^{2N} \left(1 - \frac{\epsilon}{2^N} \right)^2 \right)^{1/2} \\ &> 1 - \frac{\epsilon}{2^N} > \epsilon. \end{aligned}$$

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* This example is similar to one due to Birkhoff, cited indirectly on page 378 of [6].

Thus although every element of the class $S_{p'q}$ ([9], p. 474), composed of functions $x(s)$ measurable to L^q and having $\|x(s)\|^{p'}$ summable, has its $V(\phi)$ c.c., this is no longer true if $S_{p'q}$ is replaced by $\mathfrak{L}_0^{p'}(L^q)$.

9.4. Let $\{x_{ij}\}$ be a complete orthonormal sequence in L^2 arranged in a doubly infinite array, and consider $y_i(s)=x_{ij}$ on $E_{ij} = [j-1/2^i, j/2^i)$, $(j=1, 2, \cdots, 2^i)$, $y_i(0)=y_i(1)=\theta$. Let $x_n(s)=\sum_{i=1}^n y_i(s)$. Since $y_i(:)$ is in $\mathfrak{L}_0^2(L^2)$ and $((y_i))_2 \leqq 1/2^{i/2}$, we have

$$((x_n - x_m))_2 \leqq \sum_{i=m+1}^{n} 1/2^{i/2} \to 0$$

as $m, n\to\infty$, so that $\{x_n\}$ is a Cauchy sequence in $\mathfrak{L}_0^p(L^2)$, $(1\leqq p\leqq 2)$. If there were an $x(:)$ in $\mathfrak{L}_0^p(L^2)$ such that $((x_n-x))_p\to 0$, then (paragraph 3.22) it follows that $((x_n-x))_1\to 0$ and $\text{l.u.b.}_E \|\int_E x_n - \int_E x\| \to 0$ with $1/n$. But as Birkhoff has observed (this example is example 5 of [6]), there exists no $x(s)$ satisfying this last condition. Hence the space $\mathfrak{L}^p(L^2)=\mathfrak{L}_0^p(L^2)$ is not complete for $1\leqq p\leqq 2$.

If V_n is the operation from $L^{p'}$ to L^2, $(2\leqq p'\leqq\infty)$, defined by the step-function x_n, then V_n is c.c., and $V=\lim V_n$ is therefore also c.c. But V is generated by no function integrable either (Bn), (Bk), ($\mathfrak{X}$), (D1), or (D2), since the last four integrals have equivalent definitions when $\mathfrak{X}=L^2$ and each of these definitions is more general than that of Bochner.

This example also shows that for step-functions, $((x))_p$ is not topologically equivalent to $(\int_0^1\|x(s)\|^p ds)^{1/p}$. Moreover, if $X(E)=\lim_n \int_E x_n(s)ds$, then $X(E)$ is the limit of $X_n(E)=\int_E x_n(s)ds$ uniformly with respect to E, and hence is a c.a. and a.c. function from measurable sets to L^2; yet $X(E)$ has a weak derivative on at most a set of measure zero, and from a remark in §8, has no pseudo-derivative. Finally, none of the integral definitions that have been considered here serve to define the general linear or general completely continuous operation from L^2 to L^2.

Conclusion. The following questions are among those we have failed to answer.

Is the integral of a measurable integrable function weakly differentiable a.e.?

Can the hypothesis of measurability be dropped in Theorem 5.3? In Theorem 6.2?

Is the ($\mathfrak{X}$) integral equivalent to that of Birkhoff?

Are step-functions dense in $\mathfrak{L}_0^1(\mathfrak{X})$ for non-separable $\mathfrak{X}$? If so, then every integrable function defines c.c. operations from $\overline{\mathfrak{X}}$ to L^1 and from L^∞ to $\mathfrak{X}$. If the ($\mathfrak{X}$) integral is equivalent to the (Bk) integral, the answer to this question is yes, by paragraph 3.22 above and Theorem 18 of [6].

REFERENCES

1. S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

2. S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933.

3. L. M. Graves, *Riemann integration and Taylor's theorem in general analysis*, these Transactions, vol. 29 (1927), pp. 163–177.

4. S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorräumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

5. N. Dunford, *Integration in general analysis*, these Transactions, vol. 37 (1935), pp. 441–453.

6. G. Birkhoff, *Integration of functions with values in a Banach space*, these Transactions, vol. 28 (1935), pp. 357–378.

7. N. Dunford, *Integration of abstract functions*, Bulletin of the American Mathematical Society, abstract 42-3-90.

8. N. Dunford, *Integration of vector-valued functions*, Bulletin of the American Mathematical Society, abstract 43-1-21.

9. N. Dunford, *Integration and linear operations*, these Transactions, vol. 40 (1936), pp. 474–494.

10. G. Birkhoff, *Moore-Smith convergence in general topology*, Annals of Mathematics, (2), vol. 38 (1937), pp. 39–57.

11. W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen* II, Studia Mathematica, vol. 1 (1929), pp. 241–255.

12. W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen* (I), Studia Mathematica, vol. 4 (1933), pp. 33–37.

13. T. H. Hildebrandt, *On bounded linear functional operations*, these Transactions, vol. 36 (1934), pp. 868–875.

14. T. H. Hildebrandt, *Linear functional transformations in general spaces*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 186–212.

15. I. Gelfand, *Sur un lemme de la théorie des espaces linéaires*, Communications de l'Institut des Sciences Mathématiques et mécaniques de l'Université de Kharkoff et la Société Mathématique de Kharkoff, (4), vol. 13, pp. 35–40.

16. G. Fichtenholz and L. Kantorovitch, *Sur les opérations dans l'espace des fonctions bornées*, Studia Mathematica, vol. 5 (1934), pp. 69–98.

17. O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fundamenta Mathematicae, vol. 15 (1930), pp. 131–179.

18. J. A. Clarkson, *Uniformly convex spaces*, these Transactions, vol. 40 (1936), pp. 396–414.

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